MATRICES WITH ZERO TRACE

BY

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ABSTRACT

Let $M_n(F)$ denote the algebra of n-square matrices with elements in a field F. In this paper we show that if $M \in M_n(F)$ has zero trace then $M = AB-BA$ for certain A, $B \in M_n(F)$, with A nilpotent and trace $B = 0$, apart from some exceptional cases when $n = 2$ or 3. We also determine when $M = MB-BM$ for some $B \in M_n(F)$.

Let F be a field of characteristic p, p zero or prime. Let $M_n(F)$ denote the algebra of n-square matrices with elements in F. Let $(A,B) = AB - BA$ denote the commutator of matrices $A,B \in M_n(F)$. It is well known that trace $(A,B) = 0$. In 1936, Shoda [2] proved for $p = 0$ that if $M \in M_n(F)$ has zero trace then $M = (A,B)$ within $M_n(F)$. In 1957, Albert and Muckenhaupt [1] removed the restriction on p. It is of interest to ask whether in $M = (A,B)$ it is possible to choose $A,B \in M_n(F)$ so that trace $A = \text{trace}\, B = 0$. If $n \not\equiv 0 (\text{mod}\, p)$ it is trivial to see that this is always possible. For let $\alpha = n^{-1}\text{trace}\, A$, $\beta = n^{-1}\text{trace}\, B$. Then $M = (A,B) = (A - \alpha I_n, B - \beta I_n)$ where I_n is the n-square identity matrix. Here $A - \alpha I_n$ and $B - \beta I_n$ each have zero trace. However, this argument fails if $n \equiv 0\ (\text{mod}\, p)$. It is still easy to see that we can always choose A to have trace zero. For if trace $B = 0$ then $M = (-B, A)$ and $-B$ has zero trace. If trace $B \neq 0$, let $\gamma = -(\text{trace}\ A)(\text{trace}\, B)^{-1}$. Then $M = (A + \gamma B, B)$ and here trace $(A + \gamma B) = 0$. No simple argument of this kind can show that it is always possible to choose both A and B to have zero trace, since we shall exhibit below an example where this is impossible. We are now ready to state our main result.

THEOREM 1. *If $p \neq 3$ let $n > 2$ and if $p = 3$ let $n > 3$. Let $M \in M_n(F)$ have zero trace. Then $A,B \in M_n(F)$ exist such that $M = (A,B)$, A is nilpotent, and B has zero trace.*

In Theorems 2, 3, 4, we supply a discussion of the cases $n = 2$ and $n = p = 3$. In Theorem 5 we obtain some consequences of Theorem 1. In Theorem 6 we determine when $M = (M,B)$ within $M_n(F)$.

We first require a Lemma that extends somewhat a Lemma proved in [1].

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LEMMA. *Let* $M=(m_{ij})\in M_n(F)$, *where* $n\geqq 2$. *Suppose*

$$\sum_{i=1}^{n-\alpha} m_{i,i+\alpha}=0,\quad 0\leqq\alpha\leqq n-1. \tag{1}$$

Let $K=(k_{ij})\in M_n(F)$ *where* $k_{ij}=0$ *if* $i\neq j+1$ *and* $k_{j+1,j}=1$ *for* $1\leqq j<n$. *Then* $B\in M_n(F)$ *exists such that* $M=(K,B)$, *with*

$$\operatorname{trace} B = -\sum_{i=1}^{n-1}(n-i)m_{i+1,i}. \tag{2}$$

Proof. Let $B=(b_{ij})$ where $b_{i1}=0$ for $1\leqq i\leqq n$. The elements in the top row and in the first column of KB are all zero, and for $\alpha>1$, $\beta>1$, the element in position (α,β) of KB is $b_{\alpha-1,\beta}$. In $-BK$ the last column is a zero column, and for $\beta<n$, the element in position (α,β) is $-b_{\alpha,\beta+1}$. Thus in column β of $KB-BK$, for $\beta<n$, we see new unknowns $b_{1,\beta+1},b_{2,\beta+1},\cdots,b_{n,\beta+1}$ that do not appear in any column of $KB-BK$ to the left of column β. We may therefore choose B such that $M-(KB-BK)$ has all columns zero, except perhaps for column n. We now introduce some additional terminology. In an n-square matrix let diagonal α denote the diagonal of positions $(i,i+\alpha-1)$, $1\leqq i\leqq n-\alpha+1$; $1\leqq\alpha\leqq n$. In $KB-BK$ diagonal n has a single element, zero, and this is also true of M. For $1\leqq\alpha\leqq n-1$, the sum down diagonal α in $KB-BK$ is

$$\sum_{i=2}^{n-\alpha+1} b_{i-1,i+\alpha-1} - \sum_{i=1}^{n-\alpha} b_{i,i+\alpha} = 0.$$

Hence the sum down diagonal α in $M-(K,B)$ is zero, $1\leqq\alpha\leqq n$. Since $M-(K,B)$ can have nonzero elements only in column n, we must have $M-(K,B)=0$. The elements $b_{11}=0,b_{22},\cdots,b_{nn}$ in B satisfy the equations:

$$m_{21} = -b_{22},$$
$$m_{i,i-1} = b_{i-1,i-1}-b_{ii},\quad 3\leqq i\leqq n.$$

Hence

$$b_{ii} = -(m_{21}+m_{32}+\cdots+m_{i,i-1}), \tag{3}$$

for $2\leqq i\leqq n$. From (3) it is easy to get (2).

We now give the proof of Theorem 1. First observe that, given $M=(m_{ij})\in M_n(F)$ with trace $M=0$, it suffices to prove Theorem 1 for some similarity transform SMS^{-1} by a nonsingular element S of $M_n(F)$. Next observe that if $D=\operatorname{diag}(d_1,d_2,\cdots,d_n)\in M_n(F)$ and is nonsingular, then the second diagonal of $D^{-1}MD$ is $d_1^{-1}m_{12}d_2,d_2^{-1}m_{23}d_3,\cdots,d_{n-1}^{-1}m_{n-1,n}d_n$. From this it follows that for appropriate nonzero $d_1,d_2,\cdots,d_n\in F$, we can in $D^{-1}MD$ replace the nonzero elements on the second diagonal of M with any given nonzero values from F. Moreover, the positions in M which are zero still are zero in $D^{-1}MD$.

We let $C(p(\lambda))$ denote the companion matrix of polynomial $p(\lambda)$. We take our companion matrices so that (when degree $p(\lambda) > 1$) the stripe of ones is on the second diagonal of $C(p(\lambda))$.

Now let

$$M = C(p_1(\lambda)) \dotplus C(p_2(\lambda)) \dotplus \cdots \dotplus C(p_r(\lambda)) \in M_n(F), \tag{4}$$

where $\dotplus$ denotes direct sum. We arrange matters so that

$$p_{i+1}(\lambda) \text{ divides } p_i(\lambda), \ 1 \leqq i \leqq r, \tag{5}$$

(possibly $r = 1$). Let d be the number of ones on the second diagonal of M.

We now suppose $F \neq GF(2)$, the two element field. A separate proof will be given later when $F = GF(2)$.

We break our discussion into cases. First let degree $p_1(\lambda) \geqq 4$ or degree $p_1(\lambda) = 3$, $r > 1$, degree $p_2(\lambda) > 1$. Then $d \geqq 3$. Select $y \in F$ such that $y \neq 0$, $y \neq -(d-3)$. This is possible if F has at least three elements. Set $x = -y-(d-3)$. Then $x \neq 0$. Find a diagonal matrix $D \in M_n(F)$ so that the nonzero elements on the second diagonal of $D^{-1}MD$ are 1, x, y together with $d-3$ ones. Let

$$D^{-1}MD = \begin{bmatrix} 0 & u \\ v & M_1 \end{bmatrix};$$

where $M_1 \in M_{n-1}(F)$; $u = (1,0,0,\cdots,0)$ is a row $(n-1)$-tuple; v is a column $(n-1)$-tuple for which the transpose, v^T, has the form $v^T = (0, v_3, v_4, \cdots, v_n)$. Owing to the choice of x and y, the sum down the second diagonal of M_1 is zero. Hence, by the Lemma, $M_1 = (K_1, B_1)$ for a certain $(n-1)$-square K_1 given by the lemma and for some $B_1 \in M_{n-1}(F)$. Set

$$A = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \end{bmatrix}, \quad B = \begin{bmatrix} -\operatorname{tr} B_1 & u_1 \\ v_1 & B_1 \end{bmatrix}. \tag{6}$$

Here $u_1 = (0,-1,0,0,\cdots,0)$, $v_1^T = (v_3, v_4, \cdots, v_n, 0)$. Then $-u_1K_1 = u, K_1v_1 = v$, and hence $D^{-1}MD = (A,B)$. Moreover A is nilpotent and trace $B = 0$.

We now have to examine the following cases: (i) degree $p_1(\lambda) = 3$, degree $p_2(\lambda) = \cdots = \text{degree } p_r(\lambda) = 1$ (perhaps $r = 1$); (ii) degree $p_1(\lambda) = 2$; (iii) degree $p_1(\lambda) = 1$.

Case (i). If $n \not\equiv 0 (\operatorname{mod} p)$, set $x = 0$. If $n \equiv 0 (\operatorname{mod} p)$ but $p \neq 3$, let x be the solution in F of $3x = 2a_2$, where $p_1(\lambda) = \lambda^3 - a_3\lambda^2 - a_2\lambda - a_1$. Defer for a moment the possibility $p = 3$, $n \equiv 0 (\operatorname{mod} 3)$. Let

$$\Delta = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{bmatrix} \dotplus I_{n-3}.$$

Then the sum down the second diagonal of $\Delta M\Delta^{-1}$ is zero. We can apply the lemma to $\Delta M\Delta^{-1}$ to get $\Delta M\Delta^{-1}=(K,B)$. If $n\not\equiv 0 \pmod p$ we also have $\Delta M\Delta^{-1}=(K,B-\beta I_n)$. If we put $\beta=n^{-1}\,\mathrm{trace}\,B$ then we have K nilpotent and $\mathrm{trace}(B-\beta I_n)=0$. If $n\equiv 0 \pmod p$ then the formula (2) together with the choice of x shows trace $B=0$. This finishes case (i), except when $p=3$ and $n\equiv 0 \pmod 3$.

When $p=3$ and $n\equiv 0 \pmod 3$, the conditions in the theorem show that $n>3$. Moreover (5) and degree $p_2(\lambda)=1$ show that $M=C(p_1(\lambda))\dotplus\gamma I_{n-3}$ for some $\gamma\in F$. But then M is similar to $M_1=\gamma I_1\dotplus C(p_1(\lambda))\dotplus\gamma I_{n-4}$. Let $D=(1)\dotplus(-1)\dotplus I_{n-2}$. Then the sum down the second diagonal of $D^{-1}M_1D$ is zero. If we apply the Lemma to $D^{-1}M_1D$ we get $D^{-1}M_1D=(K,B)$. The formula (2) for trace B (use $n=0$ in F) shows that $\mathrm{trace}\,B=0$. This completes case (i).

To handle the case in which degree $p_1(\lambda)=2$, we let

$$T_m=\begin{bmatrix}\alpha_1&\beta_1\\\gamma_1&\delta_1\end{bmatrix}\dotplus\begin{bmatrix}\alpha_2&\beta_2\\\gamma_2&\delta_2\end{bmatrix}\dotplus\cdots\dotplus\begin{bmatrix}\alpha_m&\beta_m\\\gamma_m&\delta_m\end{bmatrix}.$$

T_m is $2m$-square. We permute the rows and columns of T_m in the same way — this is a similarity transformation $P^{-1}T_mP$ of T_m by a permutation matrix P. We take the rows (and columns) of T_m in the order $1,3,5,\cdots,2m-1,2,4,6,\cdots,2m$. The result of this similarity is (in partitioned form)

$$T'_m=P^{-1}T_mP=\begin{bmatrix}\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_m), & \mathrm{diag}(\beta_1,\beta_2,\cdots,\beta_m)\\ \mathrm{diag}(\gamma_1,\gamma_2,\cdots,\gamma_m), & \mathrm{diag}(\delta_1,\delta_2,\cdots,\delta_m)\end{bmatrix}.$$

We now consider case (ii). If degree $p_1(\lambda)=\mathrm{degree}\,p_2(\lambda)=2$, we may find a diagonal D such that the second diagonal of $D^{-}MD$ sums to zero. But $D^{-1}MD=T_m$ or $D^{-1}MD=T_m\dotplus(\gamma)$ according as n is even or odd. So we find a nonsingular $Q\in M_n(F)$ such that $Q^{-1}MQ=T'_m$ or $Q^{-1}MQ=T'_m\dotplus(\gamma)$, as n is even or odd. By the Lemma, $Q^{-1}MQ=(K,B)$. Here (2) and $m>1$ show that $\mathrm{trace}\,B=0$. If $\mathrm{degree}\,p_2(\lambda)=1$ then $p_2(\lambda)=\cdots=p_r(\lambda)=\lambda-\gamma$, so $p_1(\lambda)=(\lambda-\gamma)(\lambda-\delta)$, for certain $\gamma,\delta\in F$. But then M is similar to $M_1=(\delta I_1\dotplus\gamma I_{n-1})+E_{n1}$ where E_{n1} is n-square with all entries zero except for a single one at the $(n,1)$-position. Since $n>2$, the Lemma shows $M_1=(K,B)$ where, by (2), trace $B=0$ This completes case (ii).

In case (iii), M is diagonal and by the Lemma $M=(K,B)$ with $\mathrm{trace}\,B=0$. This completes the proof of Theorem 1 when $F\neq GF(2)$.

Now assume $F=GF(2)$. Let M be given by (4) and (5). First suppose degree $p_1(\lambda)\geqq 3$. Let $M=(m_{ij})$ and consider first the case in which the number of ones on the second diagonal of M is even. Let

$$\delta=\sum_{i=1}^{n-1}m_{i+1,i}(n-i).$$

Let $s = \text{degree}\, p_1(\lambda)$, so that $C(p_1(\lambda))$ is s-square. Let $E_{s,s-2}$ be s-square with all entries zero except for a single one at position $(s, s-2)$. Let $\Delta = I_s + \delta E_{s,s-2}$. Then

$$M' = \Delta C(p_1(\lambda))\Delta^{-1} \dotplus C(p_2(\lambda)) \dotplus \cdots \dotplus C(p_r(\lambda))$$

still has an even number of ones on the second diagonal. By the Lemma $M' = (K, B)$ and by (2), $\text{trace}\, B = \delta + (n - s + 1)\delta + (n-s+2)\delta = 2(n-s+2)\delta = 0$. Now let the number of ones on the second diagonal of M be odd. Let

$$M = \begin{bmatrix} 0 & u \\ v & M_1 \end{bmatrix} \tag{7}$$

where $u = (1,0,0,\cdots,0)$, $v^T = (0, v_3, \cdots, v_n)$, and M_1 has an even number of ones on the second diagonal. Then, by the Lemma, $M_1 = (K_1, B_1)$. Define A, B by (6). Then $M = (A, B), A$ is nilpotent, $\text{trace}\, B = 0$.

We may now assume that degree $p_1(\lambda)$ is two or one. If degree $p_1(\lambda)$ is one, then M is diagonal and the Lemma applies to M to give the result. So let degree $p_1(\lambda)$ be two. Then $p_1(\lambda)$ is one of $\lambda^2, \lambda^2 + \lambda, \lambda^2 + 1, \lambda^2 + \lambda + 1$. If $p_1(\lambda) = \lambda^2$, then if there are an even number of ones on the second diagonal of M the Lemma immediately gives the result. If there are an odd number of ones on the second diagonal then M is given by (7) with $v = 0$. Then, by the Lemma, $M_1 = (K_1, B_1)$. (M_1 has at least two rows since M has at least three rows.) Let A, B be given by (6), with $v_1 = 0$. Then $M = (A, B)$ with A nilpotent and $\text{trace}\, B = 0$. If $p_1(\lambda) = \lambda^2 + \lambda$ then (because of (5)), M is diagonable and the result is at hand. If $p_1(\lambda) = \lambda^2 + 1$ then M is similar to

$$M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \dotplus \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \dotplus I_{n-2s},$$

where there are s copies of

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

If $s > 1$, then M_1 has the form $M_1 = T_m$ or $M_1 = T_m \dotplus (1)$, according as n is even or odd, with $\beta_1 = \cdots = \beta_m = 0$. But then there exists Q such that $Q^{-1}M_1Q = T_m$ or $Q^{-1}MQ = T_m' \dotplus (1)$. By the Lemma, (2) and $m > 1$, $Q^{-1}M_1Q = (K, B)$ with trace $B = 0$. If $s = 1, M$ is similar to $I_n + E_{n1}$, and by the Lemma, $I_n + E_{n1} = (K, B)$, with trace $B = 0$. We now have to consider the case $p_1(\lambda) = \lambda^2 + \lambda + 1$. Then, as $p_1(\lambda)$ is irreducible, $p_1(\lambda) = p_2(\lambda) = \cdots = p_r(\lambda)$ and $\text{trace}\, M = r$. Thus r is even. But then the sum down the second diagonal of M is zero. Moreover $M = T_r$. So M is similar to T_r' and by the Lemma $T_r' = (K, B)$ with trace $B = 0$. This completes the proof of Theorem 1.

THEOREM 2. *Let $p = 3$. Let $M \in M_3(F)$, with trace $M = 0$. Then:* (i) *$M = (A,B)$ within $M_3(F)$ with A nilpotent and trace $B = 0$ if and only if the characteristic polynomial $p(\lambda)$ of M has the form*

$$p(\lambda) = \lambda^3 - x^2\lambda - \delta, \quad x,\delta,\in F; \tag{8}$$

(ii) *$M = (A,B)$ within $M_3(F)$ with A nilpotent;* (iii) *$M = (A,B)$ within $M_3(F)$ with trace $A =$ trace $B = 0$.*

Proof. Suppose $M = (A,B)$ within $M_3(F)$ with A nilpotent and trace $B = 0$. After a similarity transformation of $M = (A,B)$ by a nonsingular element of $M_3(F)$, we may assume A is one of the following three matrices:

$$A = 0; \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{9}$$

If $A = 0$ then $M = 0$ and the characteristic polynomial of M has the form (8). From $M = (A,B)$ we get $M = (A, B - \beta I_3)$ and trace $(B - \beta I_3) =$ trace B for any $\beta \in F$. So in $M = (A,B)$ we may assume that the $(3,3)$ element of B is zero. Hence let

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & -b_{11} & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix}. \tag{10}$$

If we compute the characteristic polynomial of (A,B) where A is the second matrix (9) and B is given by (10), we get that the coefficient of λ is $-b_{23}^2$. If we compute the characteristic polyniomal of (A,B) where A is the third matrix (9) and B is given by (10), we get (using $2 = -1$ in F) that the coefficient of λ is $-(b_{12} + b_{23})^2$. Hence the characteristic polynomial of (A,B) has the form (8).

Suppose now the characteristic polynomial $p(\lambda)$ of M is given by (8). If M is nonderogatory then M is similar to $C(p(\lambda))$. But $C(p(\lambda)) = (U,V)$ where

$$U = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ x & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -x & -1 \\ \delta & x^2 & x \\ -\delta x & -x^3 & -x^2 \end{bmatrix}.$$

Here U is nilpotent and V has trace zero. Suppose M is derogatory. Then $p(\lambda)$ must have a repeated root. Let γ, γ, α be the roots of $p(\lambda)$. Then $\gamma + \gamma + \alpha = 0$ and $\gamma + \gamma + \gamma = 0$ (since F has characteristic 3). Thus $\alpha = \gamma$. Hence $p(\lambda) = (\lambda - \gamma)^3$. As M is derogatory the minimal polynomial of M must be $\lambda - \gamma$ or $(\lambda - \gamma)^2$ and, of course, the minimal polynomial has coefficients in F. Thus $\gamma \in F$ and M is similar within $M_3(F)$ to

$$\begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ \varepsilon & 0 & \gamma \end{bmatrix} \tag{11}$$

where ε is 0 or 1. But by the Lemma, for M given by (11), $M = (K,B)$ where, using (2), trace $B = 0$. This proves (i).

To prove (ii), first let M be nonderogatory, similar to $C(g(\lambda))$ for some polynomial $g(\lambda)$. Choose diagonal D such that the second diagonal of $D^{-1}C(g(\lambda))D$ sums to zero. Then by the Lemma, $D^{-1}C(g(\lambda))D = (K,B)$ where K is nilpotent. If M is derogatory then the argument given above shows M is similar within $M_3(F)$ to the matrix (11). Hence always $M = (A,B)$ where A is nilpotent. And in fact we have proved that if M is derogatory then $M = (A,B)$ with A nilpotent and trace $B = 0$, within $M_3(F)$. To prove (iii) therefore we may assume $M = C(g(\lambda)$. Let $g(\lambda) = \lambda^3 - \alpha\lambda - \beta$. Let now $U = \text{diag}(0,1,-1)$,

$$V = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -\beta & \alpha & 0 \end{bmatrix}.$$

Then $M = (U,V)$ and trace $U = \text{trace}\, V = 0$. (Use $2 = -1$ in F.) This completes the proof of Theorem 2.

THEOREM 3. *Let $p \neq 2$ and let $M \in M_2(F)$ with trace $M = 0$.* (i) *If $M = (A,B)$ within $M_2(F)$ with A nilpotent then the eigenvalues of M are in F. If the eigenvalues of M are in F then $M = (A,B)$ within $M_2(F)$ with A nilpotent and trace $B = 0$.* (ii) *$M = (A,B)$ within $M_2(F)$ with trace A = trace B = 0 can always be achieved.*

THEOREM 4. *Let $p = 2$ and let $M \in M_2(F)$ with trace $M = 0$.* (i) *$M = (A,B)$ within $M_2(F)$ with A nilpotent if and only if the eigenvalues of M are in F.* (ii) *If $M = (A,B)$ within $M_2(F)$ with trace A = trace $B = 0$ then M is scalar. If M is scalar then $M = (A,B)$ within $M_2(F)$ with both A, B nilpotent.* (iii) *$M = (A,B)$ within $M_2(F)$ with trace $A = 0$ can always be achieved.*

Proofs. Let $M = (A,B)$ with A nilpotent. Either $A = 0$ (and then $M = 0$) or, after a similarity transformation by a nonsingular element of $M_2(F)$, we may assume

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $B = (b_{ij})_{1 \leq i,j \leq 2}$. Then

$$(A,B) = \begin{bmatrix} -b_{12} & 0 \\ b_{11} - b_{22} & b_{12} \end{bmatrix}.$$

Hence the eigenvalues of (A, B) are in F. Conversely if the eigenvalues of $M = (m_{ij})$ are in F, after a similarity transformation we may assume $m_{12} = 0$. Then $M = (A,B)$ where $A = E_{21}$ and

$$B = \begin{bmatrix} m_{21} & -m_{11} \\ 0 & 0 \end{bmatrix}$$

If $p \neq 2$, we also have $M = (A, B - 2^{-1} m_{21} I_2)$ and $\text{trace}(B - 2^{-1} m_{21} I_2) = 0$. This proves part (i) of each theorem.

Let $p = 2$ and let $M = (A,B)$ with $\text{trace}\, A = \text{trace}\ B = 0$. Then also $M = (A - \alpha I, B - \beta I)$ with α equal to the (2,2) element of A, and β equal to the (2,2) element of B, and $\text{trace}(A - \alpha I) = \text{trace}(B - \beta I) = 0$. So in $M = (A,B)$ we may assume the main diagonal is zero. Then

$$\left(\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix} \right)$$

is scalar. On the other hand if $M = mI_2$, then

$$M = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \right)$$

is the commutator of two nilpotent matrices. This proves Theorem 4(ii).

To prove Theorem 4(iii) we may assume M is not scalar. In Theorem 3(ii) a nonzero M with trace zero cannot be scalar. So to complete these proofs let $M = C(\lambda^2 - a)$. Then $M = (A,B)$ where

$$A = \begin{bmatrix} 0 & -1 \\ a & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $p \neq 2$ then also $M = (A, B - 2^{-1} I_2)$ and $\text{trace}\ (B - 2^{-1} I_2) = 0$. This finishes the proofs of Theorems 3 and 4.

THEOREM 5. *Let $M \in M_n(F)$, $n > 2$, with trace $M = 0$. Then M is an arbitrary word in commutators within $M_n(F)$.*

Thus, for example, $M = ((A_1, A_2), ((A_3, A_4), A_5))$ within $M_n(F)$ if and only if trace $M = 0$.

We now require additional terminology. Let L be the algebraic closure of field F. The invariant factors of $M \in M_n(F)$ are by definition the nonconstant polynomials on the main diagonal of the Smith canonical form of the polynomial matrix $\lambda I - M$. Over F, each invariant factor of M can be split into a product of powers of irreducible polynomials over F. We call these powers of irreducible polynomials over F the elementary divisors of M over F. Over L, each elementary divisor has the form $(\lambda - \lambda o)^m$.

THEOREM 6. *Let $M \in M_n(F)$. Then $B \in M_n(F)$ exists such that $M = (M, B)$ if and only if each elementary divisor $(\lambda - \lambda_0)^m$ of M over L has $m \equiv 0 \pmod p$ whenever $\lambda_0 \neq 0$. If this condition is satisfied then it is always possible to choose B such that* trace $B = 0$, *except in one situation: if $p = 2$ and if each elementary divisor of M over L has even degree, then for all choices of B we have* trace $B = n/2$.

An equivalent form of the condition of Theorem 6 is that each elementary divisor of M over F not of the form λ^m be a polynomial over F in λ^p.

Proof. Suppose $M = (M, B)$. After a similarity transformation by a nonsingular element of $M_n(L)$, we may suppose $M = M_1 \dotplus \cdots \dotplus M_r$, where M_i is m_i-square, of the form $M_i = (\lambda_i)$ if $m_i = 1$, or

$$M_i = \lambda_i I_{m_i} + C(\lambda^{m_i})$$

if $m_i > 1$. (Jordan canonical form.) Here the λ_i are not necessarily different. Partition $B = (B_{ij})$ $1 \leqq i,j \leqq r$, where B_{ii} is m_i-square. Then $M = (M, B)$ implies $M_i = (M_i, B_{ii})$, $1 \leqq i \leqq r$. Hence trace $M_i = 0$. This implies that $m_i \equiv 0 \pmod p$ whenever $\lambda_i \neq 0$. Hence the condition of Theorem 6 is satisfied. Suppose now that $p = 2$ and that each m_i is even. Fix i, and let $B_{ii} = (b_{\alpha\beta})$. Then $M_i = (M_i, B_{ii})$ yields $b_{\alpha+1,\alpha+1} - b_{\alpha\alpha} = 1$, for $1 \leqq \alpha < m_i$. Hence $b_{\alpha+1,\alpha+1} = \alpha + b_{11}$, and hence trace $B_{ii} = m_i(m_i - 1)/2 + m_i b_{11} = m_i/2$ because $m_i = 0$ in L. Therefore trace $B = (m_i + \cdots + m_r)/2 = n/2$.

To complete the proof of Theorem 6, we suppose $M \in M_n(F)$ satisfies the condition of Theorem 6. We have to find $B \in M_n(F)$ such that $M = (M, B)$, with trace $B = 0$, apart from the exceptional case. Let $\phi(\lambda)^e$ be an elementary divisor of M over F, with $\phi(\lambda) \neq \lambda$. Let λ_0 be a root of $\phi(\lambda)$ of multiplicity v, where $\lambda_0 \in L$. Then $(\lambda - \lambda_0)^{ve}$ is an elementary divisor of M over L. Hence either $v \equiv 0 \pmod p$ or $e \equiv 0 \pmod p$. In either event $\phi(\lambda)^e$ must be a polynomial in λ^p. Now let $g(\lambda) = -a_0 - a_1\lambda - \cdots - a_{m-1}\lambda^{m-1} + \lambda^m$ be a polynomial in λ^p: $a_j = 0$ if $j \equiv 0 \pmod p$, and $m \equiv 0 \pmod p$. Let $B_1 = \mathrm{diag}(1, 2, 3, \cdots, m)$. Then $C(g(\lambda)) = (C(g(\lambda)), B_1)$, since $(j+1)a_j = 0 = a_j$ if $j \not\equiv 0 \pmod p$, and $(j+1)a_j = a_j$ if $j \equiv 0 \pmod p$. Moreover, for odd p, trace $B = m(m+1)/2 = 0$ becuse $m \equiv 0 \pmod p$. Next note that $C(\lambda^m) = (C(\lambda^m), B_1 - \alpha I_m)$ for any m and any $\alpha \in F$. If p is odd and $m \equiv 0 \pmod p$, put $\alpha = 0$. Then trace $B_1 = 0$. If $m \equiv 0 \pmod p$, α may be chosen from F so that trace$(B_1 - \alpha I_m)$ achieves any desired value in F. By taking direct sums, we can get $M = (M, B)$ within $M_m(F)$, with trace $B = 0$ in all cases but the indicated one. This completes the proof of Theorem 6.

THEOREM 7. *Let $M \in M_n(F)$, $n > 2$, ($n > 3$ if $p = 3$) with trace $M = 0$. Then*

$$M = (((\cdots((A, C), C), \cdots), C), X) \tag{12}$$

for certain $A, B, X \in M_n(F)$ *with* $\operatorname{trace} X = 0$, A *nilpotent, and* (*for* $p \neq 2$), $\operatorname{trace} C = 0$.

Proof. By Theorem 1, $M = (A, X)$ with A nilpotent and $\operatorname{trace} X = 0$. By Theorem 6, $A = (A, C)$, with $\operatorname{trace} C = 0$ for $p \neq 2$. By iteration we get (12).

References

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